Time Interval Operators

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In this work we will advance farther along a line previously developed concerning our proposal of a time interval operator, by working out the consequences of a hypothesis about the time evolution operator. The time interval operator is Hermitian and has some peculiar properties. With the help of the Discrete Phase Space Formalism (DPSF) previously developed, we are able to identify an algebraic structure within the time evolution formalism, when particular sets of discrete time intervals are considered. From that, a simple system is proposed as a quantum clock .

I. INTRODUCTION

In quantum mechanics the concept of time has been an issue always submitted to continuous scrutiny and discussion, specially if one aims to treat it as an observable. Although very interesting results have been already achieved, the question is still far from to be closed. As it was shown long ago by Pauli [1], direct attempts to find an operator which obeys a canonical commutation relation with the Hamiltonian and, at the same time, that have a reasonable physical interpretation are bounded to fail. In fact, there are in the literature not too many proposals of Hermitian operators which can be somehow related to time.

In this connection there are interesting approaches to the question of time as an observable in quantum mechanics which should be mentioned. For example, the method which takes benefit of a doubled Hilbert space [2–5], while others explore the possibility of defining observables as positive-operator-valued (POV) measures [6]. In what refers to the intriguing concept of time in the context of barrier penetration there are several different approaches, although most of them do not concern a time operator [7–9], exception made to [10].

Notwithstanding all this interesting approaches, there seems not to be an attempt which can, at the same time, display a known algebra satisfied by the Hamiltonian and the 'time operator', with a physical interpretation for all quantities involved, and with the time operator being an orthodox Hermitian observable. From another point of view, we must also mention the work by Pegg [11], which looks for a complementary operator to the Hamiltonian, that have some results similar to those of this paper. In the following we shall develop an idea which we recently proposed [12], that fulfills all the conditions mentioned above provided that the physical systems under consideration satisfy given conditions.

The starting point of our approach is based on the consideration of quantum degrees of freedom characterized by finite dimensional state spaces. As has been shown, in these cases it is possible to make use of the Discrete Phase Space Formalism (DPSF) as a suitable auxiliary tool [13–15] since this formalism can be seen as a generalization of the Weyl-Wigner formalism to degrees of freedom without classical counterpart. As such, operators acting on the appropriate Hilbert (ket) spaces can be mapped onto functions of integer indices provided that a basis in the operator space is well defined. By its turn, it has been proposed that the functional form of the operator basis elements consist, in general terms, of a double discrete Fourier transform of the Schwinger operator basis [16]. Once the action of the operator basis elements on relevant state kets is defined, the mapping procedure, that takes operators to the corresponding representative function in the discrete phase space, can be put to practical terms. In section II we briefly review the main ideas referring to the operator basis.

Since we dispose of a mapping procedure obtained from the DPSF, operator equations can be converted into ordinary equations involving the phase space counterparts (the mapped expressions) of the respective operators. If we restrict ourselves to degrees of freedom of the type we are considering, with no classical counterpart, this convenient feature permits us to show that it is possible to find an operator, which we call time interval operator, that is a generator of energy shifts and its exponential obeys a Weyl-Schwinger commutation relation with the time evolution operator. This particular operator, defined for such systems, is a Hermitian operator such that its spectrum is a discrete set of time intervals with an important role in the dynamical evolution. This is, however, only accomplished by a certain class of Hamiltonians, and the Weyl-Schwinger algebra is fulfilled only for some prescribed time intervals, which are in fact the eigenvalues of the time interval operator. In this connection we shall see that there is no general time interval operator but rather a different time interval operator for each particular Hamiltonian which admits such a 'canonical' pair. The time evolution governed by this particular time interval operator, satisfying that algebra, characterizes a particular dynamics since it evolves the state of the system over the sites of the discrete phase space occupying only one site (exhausting the total probability) at each time interval, and must not be confused with the general time evolution governed by the usual continuous time, which, by its turn, can always be carried out. This discussion is basically what is accomplished in sections III and IV.

Although the proposal of a discrete time has been discussed in other contexts, in this work we are not making any hypothesis about the background c-number valued parameter t, but rather discussing the properties of an operator and its eigenstates, a set of discrete time intervals which we denote 'quantum clock times', as we propose a simple quantum clock in the end of section IV. The move from the independent variable time to the concept of time interval is the key to the whole process, that in Section V is presented with some general remarks and conclusions.

II. THE OPERATOR BASIS

Considering a time independent Hamiltonian and its set of eigenstates consisting of an $N-$dimensional ket space,

$$H|u_k\rangle = E_k|u_k\rangle, \qquad k = 0, 1, ...N-1, \tag{1}$$

we construct the operator

$$V = \sum_{k=0}^{N-1} |u_k\rangle\langle u_{k+1}|, \tag{2}$$

where hereafter we adopt a cyclic notation

$$|u_k\rangle \equiv |u_{k(\mathrm{mod}\, N)}\rangle \tag{3}$$

which in particular means that in Eq.(2) the last term of the summation is in fact $|u_{N-1}\rangle\langle u_0|$. From this definition it follows that

$$V^s|u_n\rangle = |u_{n-s}\rangle. \tag{4}$$

Now the operator V can in turn be diagonalized,and once its eigenstatesare obtained, it can be directly verified that the well known Schwinger unitary operators algebra results are reproduced [16], that is,

$$U|u_k\rangle = \exp\left[\frac{2\pi i}{N}k\right]|u_k\rangle, \qquad V|v_k\rangle = \exp\left[\frac{2\pi i}{N}k\right]|v_k\rangle, \tag{5}$$

$$U^N = V^N = \hat{1}, \tag{6}$$

$$V^s|u_n\rangle = |u_{n-s}\rangle, \qquad U^s|v_n\rangle = |v_{n+s}\rangle, \tag{7}$$

being that in our case the set $\{|u_n\rangle\}$ diagonalizes both the Hamiltonian and the Schwinger operator U. Furthermore, the two sets of states are connected by a discrete Fourier transform

$$\langle v_k|u_n\rangle = \frac{1}{\sqrt{N}}\exp\left[-\frac{2\pi i}{N}kn\right], \tag{8}$$

and the operators obey the Weyl-Schwinger algebra

$$U^j V^l = \exp\left[\frac{2\pi i}{N} jl\right] V^l U^j. \tag{9}$$

It has been also shown that the set of N^2 operators [15]

$$\hat{G}(m,n) = \frac{1}{N} \sum_{j,l=-\frac{N-1}{2}}^{\frac{N-1}{2}} U^j V^l \exp\left[\frac{\pi i}{N} jl\right] \exp\left[-\frac{2\pi i}{N}(mj+nl)\right] \tag{10}$$

constitutes a complete and orthogonal basis in operator space; the relevant properties of this basis are collected in reference [15], from which we recall the main results. In the form that it is presented, this basis is suited to deal with odd dimensional spaces, although it can be easily adapted to even ones. In fact, as this basis keeps, from the Schwinger basis, the property of decomposition into independent sub-bases, each with a prime dimension [16], we, for simplicity, shall hereafter consider only prime N's, although most of the results in what follows do not depend on this condition. We are also omitting the so called modular phase, which should, in principle at least, be present in the basis, but is irrelevant in the particular present calculations.

Thus, any operator acting on those finite dimensional state spaces can be written in the form

$$\hat{O} = \frac{1}{N} \sum_{j,l=-\frac{N-1}{2}}^{\frac{N-1}{2}} o(m,n) \hat{G}(m,n), \tag{11}$$

where the coefficients (the operator representative on the discrete phase space, or equivalently, the discrete Weyl-Wigner mapped function associated to the operator) are given by

$$o(m,n) = Tr[\hat{G}^{\dagger}(m,n)\hat{O}]. \tag{12}$$

III. TIME EVOLUTION AS A LATTICE SHIFT

Systems with a time independent Hamiltonian H have a time evolution operator of the form

$$\mathcal{G}(\Delta t) = \exp\left[\frac{-iH\Delta t}{\hbar}\right], \tag{13}$$

that propagates the states $|\psi(t)\rangle$, to which corresponds the density operator $\hat{P}(t) = |\psi(t)\rangle\langle\psi(t)|$ for a general pure state (although this is not an essential assumption for what follows) in such a way that

$$\hat{P}(t+\Delta t) = \mathcal{G}(\Delta t)\hat{P}(t)\mathcal{G}^{\dagger}(\Delta t). \tag{14}$$

Now let us map this operator equation onto the discrete phase space assuming the *hypothesis*

$$\mathcal{G}(\Delta\tau) = U^{-k}, \quad k \in [0, N-1], \tag{15}$$

that is, the unitary time evolution operator is equal to some integer power of the Schwinger unitary operator U, at least for some given time interval $\Delta\tau$. A negative power is chosen as a matter of convenience, recalling that, according to Eq. (6), $U^{-k} = U^{N-k}$. Fairness of such hypothesis shall be justified *a posteriori*.

Hypothesis (15) has a direct consequence. If it holds for a given $\Delta\tau$, then it follows that, for any integer n,

$$\left(\exp\left[-\frac{iH\Delta\tau}{\hbar}\right]\right)^n = \left(U^{-k}\right)^n \tag{16}$$

$$\exp\left[-\frac{iHn\Delta\tau}{\hbar}\right] = U^{-kn}, \tag{17}$$

and cyclicity of U implies that

$$\exp\left[-\frac{iHn\Delta\tau}{\hbar}\right] = U^{-kn \pmod{N}}, \tag{18}$$

which obviously means that if Eq. (15) holds for a given time interval, a similar equation must hold for all its integer multiples. In view of that we can rewrite our hypothesis as

$$\mathcal{G}(\Delta\tau_n) = U^{-k'}, \quad k' \in [0, N-1], \tag{19}$$

where

$$\Delta\tau_n = n\Delta\tau, \quad n = 1, 2, 3..., \tag{20}$$

and

$$k' = kn \pmod N, \tag{21}$$

what means that hypothesis (15) refers in fact to an infinite sequence of time intervals and not just to a given one.

A less straightforward consequence can be extracted with the help of the discrete phase space representation of the density operator which is, as in the continuous cases, called Wigner function and denoted by $\rho_w(m, n; t)$. This function can be immediately calculated from

$$\rho_w(m, n; t + \Delta\tau) = Tr[G^\dagger(m, n)\hat{P}(t + \Delta\tau)], \tag{22}$$

and if one assumes that the hypothesis (15) is true, then (using Eq. (14))

$$\rho_w(m, n; t + \Delta\tau) = Tr[G^\dagger(m, n)U^{-k}\hat{P}(t)U^k]. \tag{23}$$

Now, if we substitute the density operator at time t by the expression giving its decomposition in the operator basis,

$$\rho_w(m, n; t + \Delta\tau) = Tr[\hat{G}^\dagger(m, n)U^{-k}\left(\frac{1}{N}\sum_{r,s=-h}^{h}\rho_w(r, s; t)\hat{G}(r, s)\right)U^k] \tag{24}$$

$$\rho_w(m, n; t + \Delta\tau) = Tr[\sum_{j,l=-h}^{h}\frac{1}{N}V^{-l}U^{-j}\exp\left[-\frac{i\pi}{N}jl\right]\exp\left[\frac{2\pi i}{N}(mj + nl)\right]\times$$

$$U^{-k}\left(\frac{1}{N}\sum_{r,s=-h}^{h}\rho_w(r, s; t)\sum_{x,z=-h}^{h}\frac{1}{N}U^zV^x\exp\left[\frac{i\pi}{N}xz\right]\exp\left[-\frac{2\pi i}{N}(xr + zs)\right]\right)U^k]. \tag{25}$$

Using the Weyl commutation relation between V^x and U^k,

$$\rho_w(m, n; t + \Delta\tau) = \frac{1}{N^3}\sum_{j,l=-h}^{h}\sum_{r,s=-h}^{h}\sum_{x,z=-h}^{h}Tr[V^{-l}U^{-j}U^zV^x]\exp\left[-\frac{i\pi}{N}jl\right]\exp\left[\frac{2\pi i}{N}(mj + nl)\right]$$

$$\exp\left[-\frac{2\pi i}{N}xk\right]\rho_w(r,s;t)\exp\left[\frac{i\pi}{N}xz\right]\exp\left[-\frac{2\pi i}{N}(xr+zs)\right]. \tag{26}$$

The trace contribution on the r.h.s. of (26) is $N\delta^{[N]}_{z,j}\delta^{[N]}_{x,l}$, where $\delta^{[N]}_{a,b}$ is a Kronecker delta modulo N (*i.e.*, it is only different from zero if $n = i \pmod N$), and then

$$\rho_w(m,n;t+\Delta\tau) = \frac{1}{N^2}\sum_{j,l=-h}^{h}\sum_{r,s=-h}^{h}\exp\left[\frac{2\pi i}{N}(j(m-r)+l(n-s-k))\right]\rho_w(r,s;t). \tag{27}$$

We first observe that the sums over $\{j,l\}$ also yield Kronecker deltas $\bmod N$, in fact $N\delta^{[N]}_{m,r}\delta^{[N]}_{n-k,s}$, and after the sums over $\{r,s\}$ are performed we finally obtain

$$\rho_w(m,n;t+\Delta\tau) = \rho_w(m,n-k;t). \tag{28}$$

Equation (28) shows that hypothesis (15), for that precise time interval for which it holds, leads the system state to a situation equivalent to a shift of k sites along one direction in the two-dimensional discrete phase space. Use of equation (19) characterizes a time evolution, since for each time interval there will be associated a given shift in phase space. A particularly simple and elucidative physical situation where this occurs is presented in reference [12].

We are yet to determine the physical conditions which allow our hypothesis to be valid. In order to achieve that we make use of the discrete phase space representation of the operator equation proposed as our hypothesis. The mapped expressions of the individual operators read

$$(U^{-k})(m,n) = \exp\left[-\frac{2\pi i}{N}km\right], \tag{29}$$

$$(\mathcal{G}(\Delta t))(m,n) = g(m,n;\Delta t) = \exp\left[-\frac{iE_m\Delta t}{\hbar}\right], \tag{30}$$

respectively, so that on the discrete phase space Eq. (15) will be written as

$$\exp\left[-\frac{iE_m\Delta t}{\hbar}\right] = \exp\left[-\frac{2\pi i}{N}km\right], \tag{31}$$

and therefore

$$\frac{E_m \Delta t}{\hbar} = \frac{2\pi}{N} km \pmod{2\pi}. \tag{32}$$

Equation (32) is in fact a set of N equations (one for each m) which must hold separately for a given Δt *and* for a given k. It can be still written as

$$E_m \frac{N\Delta t}{2\pi\hbar} = km + Nf(m). \tag{33}$$

where $f(m)$ is an arbitrary integer function of an integer variable. The r.h.s. of Eq. (33) is an integer number, whereas the l.h.s is a product of two, in principle, real numbers. It is then clear that a necessary but not sufficient condition to Eq. (33) holds is that there must be a real number λ for which the whole set $\left\{\frac{E_m}{\lambda}\right\}$ is an integer. If we want that condition to become sufficient, the set must be additionally a complete set of remainders modulo N [17]. Summarizing, Eq. (31) will only have solutions if and only if the spectrum can be written as

$$E_m = \hbar\omega \left(km + \ Nf(m)\right), \tag{34}$$

where ω is an appropriately defined constant. From (32) and (34) it follows that the smallest time interval for which $\mathcal{G}(\Delta t) = U^{-k}$ is

$$\Delta\tau = \frac{2\pi}{N\omega} \tag{35}$$

and for all integer multiples of $\Delta\tau$ the time evolution operator will be equal to a given power of U. Once N is a prime number, the set of the first N powers of U^{-k} will be equivalent to the set $\{1, U, U^2, .., U^{N-1}\}$ in some permuted order. Therefore, all powers of U (and consequently the respective sites of the discrete phase space) are visited as the time intervals are counted.

IV. TIME INTERVAL OPERATOR

We have noticed that, for Hamiltonians which fulfill Eq. (34) and on time intervals which are multiples of (35), the time evolution operator is equal to an integer power of the

Schwinger unitary operator U. If that is true, the Weyl commutation relation can be written as

$$\exp\left[\frac{-iH(n\Delta\tau)}{\hbar}\right]V^{-j} = \exp\left[\frac{2\pi i}{N}j(nk)\right]V^{-j}\exp\left[\frac{-iH(n\Delta\tau)}{\hbar}\right]. \tag{36}$$

On the other hand, we can write the Schwinger operator V itself as the exponential of an operator if we define

$$\mathrm{T} = \frac{2\pi}{\omega N}\sum_{j=0}^{N-1} j|v_j\rangle\langle v_j|, \tag{37}$$

as it can be directly verified that

$$\exp\left[-\frac{i\mathrm{T}(E_{m+s}-E_m)}{\hbar}\right] = V^{-ks}, \tag{38}$$

once the energy spectrum satisfies (34). Equation (36) therefore can be written as

$$\exp\left[\frac{-iHn\Delta\tau}{\hbar}\right]\exp\left[\frac{-i\mathrm{T}\Delta E_j}{\hbar}\right] = \exp\left[\frac{2\pi i}{N}nkjk\right]\exp\left[\frac{-i\mathrm{T}\Delta Ej}{\hbar}\right]\exp\left[\frac{-iHn\Delta\tau}{\hbar}\right], \tag{39}$$

where $\Delta E_j = E_{s+j} - E_s$.

From Eq. (37) it is not difficult to obtain the eigenstates and eigenvalues of T, namely

$$\mathrm{T}|v_l\rangle = t_l|v_l\rangle = \Delta\tau l|v_l\rangle, \tag{40}$$

and its phase space representative

$$(\mathrm{T})(m,n) = \Delta\tau n. \tag{41}$$

A. Quantum Clock

Let a given physical system S, described by a Hamiltonian whose spectrum obeys Eq. (34), be, at an instant t_0, in an eigenstate $|v_i\rangle$ of the time interval operator. Its corresponding initial discrete Wigner function is

$$\rho_w(m,n;t_0) = Tr[G^{\dagger}(m,n)|v_i\rangle\langle v_i|] \tag{42}$$

which can be easily calculated as

$$\rho_w(m, n; t_0) = \delta^{[N]}_{n,i}. \tag{43}$$

Using the result of Eq. (28) for the time evolution

$$\rho_w(m, n; t_0 + \Delta\tau) = \rho_w(m, n + k; t_0)$$

it is simple to verify that

$$\rho_w(m, n; t_0 + \Delta\tau) = \delta^{[N]}_{n,i+k},$$

which can be generalized, through the use of (19) to

$$\rho_w(m, n; t_0 + j\Delta\tau) = \delta^{[N]}_{n,i-jk}.$$

This is a quite interesting result. It simply states that, when the system starts its evolution at (or visit in a given instant) an eigenstate of the time interval operator, it will be in another eigenstate, with *no uncertainty,* after every time interval $\Delta\tau$. All eigenstates shall be visited in the time interval $(N-1)\Delta\tau$, although, if $k \neq 1$, they may not appear in the correct order. In general the sequence will be

$$|v_i\rangle, |v_{i+k}\rangle, |v_{i+2k}\rangle......|v_{i+(N-1)k}\rangle, \tag{44}$$

where the reader should remind of the modulo N extraction in the subscript of the ket labels. In view of that, the next ket in the above list would be

$$|v_{i+Nk}\rangle \equiv |v_i\rangle, \tag{45}$$

and the system would be back to where it started, that is, within such a cycle, the index of state ket is a 'counter' of time.

This strongly motivates us to denote this dynamically determined $\Delta\tau$ (and its first $(N-1)$ integer multiples) as quantum clock time.

V. CONCLUSIONS

The basic result that we have achieved here is a direct consequence of the quantum mechanics associated to degrees of freedom characterized by a finite dimensional state space and from trying to give quantum mechanical meaning to time intervals rather than to time itself. We have identified a Hermitian operator, a quantum mechanical observable, which is directly related to time intervals. We have denoted this operator as time interval operator, and we stress again that it is not directly identified with the continuous usual time parameter t. When equations (15) and (34) are simultaneously satisfied we are dealing with physical systems such that this operator displays the following properties:

i) it is a generator of cyclical shifts in energy state space;

ii) its exponential obeys the Weyl-Schwinger commutation relation together with the time evolution operator;

iii) it is a Hermitian operator whose eigenstates form a set of time intervals with an important role in dynamics;

iv) the eigenstates of the time interval operator are connected with the energy eigenstates through a discrete Fourier transform, therefore these two sets have a maximum degree of incompatibility.

The Weyl-Schwinger commutation relation, however, is only fulfilled for some given values of time intervals, exactly the eigenvalues of the time interval operator. The time evolution for which the Weyl-Schwinger commutation relations hold could then be called stroboscopic, in the sense that, in this case, during this time evolution, it always happens that each one of the sites of the energy sector of the discrete phase space individually will be visited and will exhaust the total probability when the time evolves as multiples of a given time interval (which are the eigenvalues of the time interval operator). By its turn, this particular dynamics must not be confused with that governed by the usual continuous time parameter, which can always be also carried out. Therefore, following the algebraic structure of those operators and their corresponding spectra, the equations characterizing the time interval operator allowed us to classify some simple physical systems that behave as a quantum clock.

In this connection, let us discuss some basic operational requirements related to the interpretation of the concept of time. To measure time, there must be a reference to a nonstationary quantity – or a property – of a physical system, and, in general, this abstract quantity admittedly changes continuously with time. From an orthodox quantum mechanical point of view, a measurement is related to a Hermitian operator, and the process of measuring may destroy the clock itself. This consideration illustrates the importance of the identification of a Hermitian operator related to time, and why we have called our time intervals of 'clock times', since a measure would only, in the worst case, 'reset' the clock. We, in fact, have not identified a continuous changing property but rather a discrete changing property of a physical system to serve as a 'quantum clock'. This comes as a natural feature of the finite dimensional spaces we are dealing with.

VI. BIBLIOGRAPHY

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[1] W. Pauli, "Die Algemeinen Prinzipien der Wellenmechanik", in *Encyclopedia of Physics*, edited by S. Flügge (Springer, 1958), Vol V, Part 1.

[2] Rosenbaum, D.M., J. Math Phys. **10** (1969) 1127.

[3] Newton, R.G., Ann. Phys. **124** (1980) 327.

[4] Bauer, M., Ann. Phys. **150** (1983) 1.

[5] Lokajíček, M. and Kundrát, V., Hadronic J. **12** (1989) 31.

[6] Busch, P., Grabowski, M. and Lahti, P.J., Phis. Lett. A **191** (1994) 357.

[7] Sokolovski, D. and Baskin, M. Phis. Rev. A. **36** (1987) 4604.

[8] Pollak, E. and Miller, W.H., Phys. Rev. Lett **53** (1984) 115.

[9] Büttiker, M. and Landauer, R., Phys. Rev. Lett **49** (1982) 1739.

[10] Kobe, D.H. and Aguilera-Navarro, V.C., Phis. Rev. A. **50** (1994) 933.

[11] Pegg, D.T., Phis. Rev. A. **58** (1998) 4307.

[12] Galetti, D. and Ruzzi, M., Physica A **264** 473.

[13] Galetti, D. and Piza, A.F.R. de Toledo, Physica (1988)

[14] Galetti, D. and Piza, A.F.R. de Toledo, Physica (1992)

[15] Ruzzi, M. and Galetti, D., J. Phys A: Math. Gen. **33** (2000) 1065. Corrigendum J. Phys A: Math. Gen. **33** (2000) 1305.

[16] Schwinger, J., "Quantum Kinematics and Dynamics" (Benjamim, N.Y., 1970).

[17] Andrews, G.E., "The Theory of Partitions", in *Encyclopedia of Mathematics and its Applications*, Ed. G.-C. Rotta, (Addison-Wesley, USA, 1976), Vol 2.